1. The magnetic vector potential $\vec{A}$

The magnetic VP is introduced as the vector whose curl equals the magnetic flux density:

$$\vec{B} = \nabla \times \vec{A}, \quad \mathrm{T}$$

Using Biot-Savart law, one can show that the magnetic VP is the potential of the electric current elements:

$$\vec{A} = \frac{\mu}{4\pi} \int_{C'} \frac{I' d\vec{l}'}{R}, \quad \frac{\mathrm{A} \times \mathrm{H}}{\mathrm{m}} = \mathrm{T} \times \mathrm{m} = \frac{\mathrm{Wb}}{\mathrm{m}}$$

Compare this with the voltage of a linear charge distribution:

$$= \frac{1}{4\pi\varepsilon} \int_{C'} \frac{\rho_l'}{R} dl', \quad \mathrm{V} = \frac{\mathrm{C} \times \mathrm{m}}{\mathrm{F}}$$

This analogy holds for all types of source distribution, including surface and volume distributions.

$$\vec{A} = \frac{\mu}{4\pi} \iint_{S'} \frac{\vec{J}'_s}{R} ds', \quad \frac{\text{Wb}}{\text{m}}$$

$$V = \frac{1}{4\pi\varepsilon} \iint_{S'} \frac{\rho'_s}{R} ds', \quad \text{V}$$

$$\vec{A} = \frac{\mu}{4\pi} \iiint_{v'} \frac{\vec{J}'}{R} dv', \quad \frac{\text{Wb}}{\text{m}}$$

$$V = \frac{1}{4\pi\varepsilon} \iiint_{v'} \frac{\rho'}{R} dv', \quad \text{V}$$

As the electric scalar potential (ESP) V is enough to fully describe the ESF, the magnetic vector potential is enough to describe the MSF. The ESP V satisfies Poisson's equation:

$$\nabla^2 V = -\rho / \varepsilon$$

In fact, the more general ESP equation (for inhomogeneous regions) is:

$$\nabla \cdot (\varepsilon \nabla V) = -\rho$$

We will now show that the magnetic VP satisfies similar equations. Starting with Ampère's law (in differential form):

$$\nabla \times \vec{H} = \vec{J}$$

and substituting: $\vec{H} = \vec{B}/\mu$,

one obtains the second-order PDE for $\vec{A}$:

$$\boxed{\nabla \times \left(\frac{1}{\mu} \nabla \times \vec{A} \right) = \vec{J}}$$

This equation is true for any (homogeneous or not) region. In the case of homogeneous regions, however, it can be simplified:

$$\nabla \times \nabla \times \vec{A} = \mu \vec{J} \qquad \Rightarrow \nabla \nabla \cdot \vec{A} - \nabla^2 \vec{A} = \mu \vec{J}$$

$\vec{A}$ is a vector field. So far, we have defined only its curl, which equals $\vec{B}$. Its divergence has to be specified, too, in order to define this vector field uniquely.

In general EM theory, the most often made choice is:

$$\nabla \cdot \vec{A} = -\mu\varepsilon \frac{\partial V}{\partial t}$$

which is called Lorentz' gauge and gives a clear link between the electric and magnetic fields as they are expressed in terms of potentials rather than field vectors. Another gauge is:

$$\nabla \cdot \vec{A} = 0$$

which is rather inconvenient in general electrodynamics. However, in the case of static fields, both gauges are identical, because $\partial V / \partial t = 0$. Then, the link between magnetic and electric field disappears. Indeed, this is the case with the ESF and the MSF. Static charges (and their field) are unaffected by permanent magnetic fields. Permanent magnets cannot register static charges. The magnetic field of steady currents is unaffected by the presence of electrostatic field. Both fields *seem* disjoint.

Lorentz' gauge does make sense from many points of view. Here, we will only point out the link between $\vec{A}$ and V, which arises because of the continuity of current law. Remember that

$$\nabla^2 \vec{A} = -\mu \vec{J} \quad \text{and} \quad \nabla^2 V = -\rho / \varepsilon$$

Take now the divergence of both sides of the left equation.

$$\nabla^2 (\nabla \cdot \vec{A}) = -\mu (\nabla \cdot \vec{J})$$

According to the continuity of current law: $\nabla \cdot \vec{J} = -\dfrac{\partial \rho}{\partial t}$

$$\Rightarrow \nabla^2 (\nabla \cdot \vec{A}) = \mu \frac{\partial \rho}{\partial t}$$

Take the time derivative from both sides of the right-hand equation, and compare the above equation with the result.

$$(-\mu\varepsilon) \nabla^2 \frac{\partial V}{\partial t} = \mu \frac{\partial \rho}{\partial t}$$

$$\Rightarrow \nabla \cdot \vec{A} = -\frac{\partial \rho}{\partial t}$$

Substituting the gauge equation, a simpler equation is obtained for the magnetic VP in homogeneous medium:

$$\nabla^2 \vec{A} = -\mu \vec{J}$$

This is Poisson's vector equation. In rectangular coordinates, this equation readily reduces to three decoupled scalar Poisson's equations for each component of the magnetic VP:

$$\nabla^2 A_x = \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} = -\mu J_x$$

$$\nabla^2 A_y = \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2} = -\mu J_y$$

$$\nabla^2 A_z = \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} = -\mu J_z$$

The magnetic VP $\vec{A}$ and the magnetic flux Ψ

$$\Psi = \iint_S \vec{B} \cdot d\vec{s} = \iint_S (\nabla \times \vec{A}) \cdot d\vec{s} = \oint_{C[S]} \vec{A} \cdot d\vec{l}$$

or

$$\boxed{\Psi = \oint_C \vec{A} \cdot d\vec{l}}, \quad \text{Wb} = \text{T} \times \text{m}^2$$

2. The analogy between ESF and MSF in the source-free case

The analogy between the electrostatic and magnetostatic field would be complete if we also answer the question: why is the potential a scalar function in the ESF case, and why is it a vectorial function in the MSF case?

The answer is rather simple: because the sources of the ESF are scalar functions of space (charge distribution), while the sources of the MSF are vectorial functions of space (current distribution).

It should be noted however that if in the region of interest there are *no currents*, the MSF can be expressed just fine with a single scalar function: the magnetic scalar potential ψ.

$$\nabla \times \vec{H} = \vec{J} = 0 \Rightarrow \nabla \times (-\nabla \psi) = 0$$

$$\Rightarrow \vec{H} = -\nabla \psi, \quad \text{A/m}$$

The analogy with the ESF is obvious: $\vec{E} = -\nabla V, \quad \text{V/m}$

The magnetic scalar potential satisfies Laplace's equation, too.

$$\nabla \cdot \vec{B} = 0 \quad \Rightarrow \quad \nabla \cdot (\mu \vec{H}) = 0 \quad \Rightarrow \quad \boxed{\nabla \cdot (\mu \nabla \psi) = 0}$$

In homogeneous regions: $\boxed{\nabla^2 \psi = 0}$

Similarly, if there are *no charges* in a given region, but there is certain ESF, it can be expressed as the curl of a vector: the electric vector potential $\vec{F}$.

$$\nabla \cdot \vec{D} = \rho = 0 \quad \Rightarrow \quad \nabla \cdot (-\nabla \times \vec{F}) = 0$$

Then, the electric flux density can be expressed as: $\vec{D} = -\nabla \times \vec{F}$

The electric VP $\vec{F}$ would also satisfy the vector Poisson's equation in homogeneous regions:

Starting from the conservative nature of the ESF:

$$\nabla \times \vec{E} = 0$$

and substituting $\vec{E} = \vec{D} / \varepsilon$,

$$\nabla \times \left(\frac{1}{\varepsilon} \vec{D} \right) = 0 \quad \Rightarrow \nabla \times \left(\frac{1}{\varepsilon} \nabla \times \vec{F} \right) = 0$$

In homogeneous regions: $\nabla \times \nabla \times \vec{F} = 0$

$$\Rightarrow \nabla \nabla \cdot \vec{F} - \nabla^2 \vec{F} = 0$$

In order to define the electric VP $\vec{F}$, we have to adopt certain gauge (to specify its divergence). In general electromagnetics, the usual gauge is again Lorentz' one, which gives a relation between the electric VP $\vec{F}$ and the magnetic scalar potential ψ .

$$\nabla \cdot \vec{F} = -\mu \varepsilon \frac{\partial \psi}{\partial t}$$

In the case of static magnetic fields, $\partial\psi/\partial t = 0$, and the gauge becomes

$$\nabla\cdot\vec{F} = 0$$

This simplifies the 2nd order PDE governing $\vec{F}$ to: $\boxed{\nabla^2\vec{F} = 0}$

3. General field equations for the ESF and the MSF: a summary

ESF

Gauss' law:

$$\nabla\cdot\vec{D} = \rho \Leftrightarrow \oiint_S \vec{D}\cdot d\vec{s} = Q$$

ESF is conservative (lamellar):

$$\nabla\times\vec{E} = 0 \Leftrightarrow \oint_C \vec{E}\cdot d\vec{l} = 0$$

MSF

Gauss' law (MSF is solenoidal):

$$\nabla\cdot\vec{B} = 0 \Leftrightarrow \oiint_S \vec{B}\cdot d\vec{s} = 0$$

Ampere's law:

$$\nabla\times\vec{H} = \vec{J} \Leftrightarrow \oint_C \vec{H}\cdot d\vec{l} = I_{tot}$$

Maxwell's equations (static case)

ESF

Coulomb's law:

$$\vec{E} = \frac{1}{4\pi\varepsilon}\frac{Q}{R^2}$$

$$= \frac{1}{4\pi\varepsilon}\frac{Q}{R}$$

$$\nabla \cdot (\varepsilon \nabla V) = -\rho$$

$$\nabla^2 V = -\rho / \varepsilon$$

$$(P) = \frac{1}{4\pi\varepsilon}\iiint_{v_Q} \frac{\rho(Q)}{R(P,Q)} dv_Q$$

MSF

Biot-Savart law:

$$\vec{B} = \frac{\mu}{4\pi}\frac{I d\vec{l} \times \hat{R}}{R^2}$$

$$\vec{A} = \frac{\mu}{4\pi}\frac{I d\vec{l}}{R}$$

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \vec{A} \right) = \vec{J}$$

$$\nabla^2 \vec{A} = -\mu \vec{J}$$

$$\vec{A}(P) = \frac{\mu}{4\pi}\iiint_{v_Q} \frac{\vec{J}(Q)}{R(P,Q)} dv_Q$$

Point source fields

Potentials and sources

ESF	MSF	
$\nabla\times\vec{E}=0\Rightarrow$ $\vec{E}=-\nabla V$	$\nabla\cdot\vec{B}=0\Rightarrow$ $\vec{B}=\nabla\times\vec{A}$	Potentials and field vectors (general)
$\nabla\cdot\vec{D}=\rho=0\Rightarrow$ $\vec{D}=-\nabla\times\vec{F}$	$\nabla\times\vec{H}=\vec{J}=0\Rightarrow$ $\vec{H}=-\nabla\psi$	Potentials and field vectors (no sources)
$\nabla\times\left(\frac{1}{\varepsilon}\nabla\times\vec{F}\right)=0$	$\nabla\cdot(\mu\nabla\psi)=0,$	
$\nabla^2\vec{F}=0,\quad \text{if}\quad \nabla\varepsilon=0$	$\nabla^2\psi=0,\quad \text{if}\quad \nabla\mu=0$	

The analogy can be continued with force, energy, and capacitance/inductance relations.